Space Filling Delaunay Loft Sculptures

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Abstract

This is a poster for a sculptural exhibition of a class of our new space filling modular shapes. This class of shapes is inspired by "scutoids" — shapes that were recently reported to occur in epithelial cells due to topological changes between the extremal (apical and basal) surfaces of epithelia. Inspired from this discovery, we develop a generalized procedure for generating space filling shapes, which we call *Delaunay Lofts* — a new class of scutoid-like shapes. Delaunay Lofts are produced as an interpolation of a stack of tilings that are defined by Delaunay diagrams. Let a stack of planar surfaces with Delaunay diagrams be given, Delaunay Lofts are the shapes that result from Voronoi tessellation of all intermediate surfaces along the curves joining the vertices of Delaunay diagrams that define the tessellations. Combined with the use of wallpaper symmetries, this process allows for an intuitive design of complex space filling shapes in 3-space.

(a) *One tile.* (b) *Two tiles.* (c) *Three tiles.* (d) *Four tiles.* (e) *Seven tiles.*

Figure 1: An example of a single Delaunay Loft tile that can fill both 2.5D and 3D space. This tile is created as an interpolation of two layers of tilings, namely (1) a square tiling; and; (2) another square tiling, which is a translation of the first square tiling. The interpolating control curves are straight lines.

1 Introduction

A space filling shape is a cellular structure whose replicas together can fill all of space watertight, i.e. without having any voids between them [11], or equivalently, it is a cellular structure that can be used to generate a tessellation of space [8]. 2D tessellations and 2D space filling shapes are relatively well-understood. However, problems related to 3D tessellations and space filling shapes are still interesting and have applications in a wide range of areas from chemistry and biology to engineering and architecture [11].

A well-known anecdote demonstrates the difficulty of 3D tessellations is that Aristotle claimed that the tetrahedron can fill space and many people tried to prove his claim [13] despite the fact that the cube is the only space filling Platonic solid [5]. Goldberg exhaustively cataloged many of known space-filling polyhedra with a series of papers from 1972 to 1982 such as [6]. There are only eight space-filling convex polyhedra and only five of them have regular faces, namely the triangular prism, hexagonal prism, cube, truncated octahedron [16, 15], and Johnson solid gyrobifastigium [10, 1]. It is also interesting that five of these eight space filling shapes are "primary" parallelohedra [3], namely cube, hexagonal prism, rhombic dodecahedron, elongated dodecahedron, and truncated octahedron.

We have recently developed an approach to construct and eventually design a new class of tilings in 3-space. Our approach is based on interpolation a stack of planar tiles whose dual tilings are Delaunay diagrams. We construct control curves that interpolate one Delaunay vertex of each planar tile. Voronoi decomposition of the volume using these control curves as Voronoi sites gives us lofted interpolation of original faces. This, combined with the use of wallpaper symmetries allows for the design of the new class of space filling shapes in 3-space. In the poster exhibition, we will demonstrate 3D printed examples of this new class of shapes (See Figures 1, 2 and, 4).

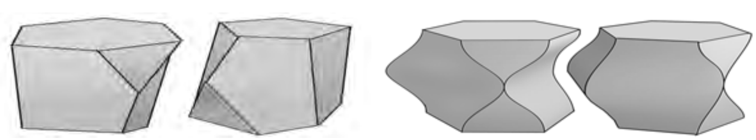
(a) *A Delaunay Loft.* (b) *Two tiles.* (c) *Three tiles.* (d) *Ten tiles.* (e) *Ten tiles.*

Figure 2: Another example of single space filling Delaunay Loft tile. This tile is created as an interpolation of three layers of tilings, namely (1) a regular hexagonal; (2) a square and; (3) another regular hexagonal tilings, which is a translation of the first hexagonal tiling. It is interesting to note that the interpolating control curves are straight lines. Regular rectangular tiling in the middle is just an automatically produced byproduct.

2 Related Work

Our approach, which can be considered as a generalization of parallelohedra, is inspired by a recent discovery by Gómez-Gálvez et al. who observed a simple polyhedral form, which they call *"scutoids"*, commonly exist in epithelial cells in the formation of thin skin layers [7]. They demonstrated that having this polyhedral form in addition to prisms provides a natural solution to three-dimensional packing of epithelial cells. In skin cells, the top (apical) and bottom (basal) surfaces of the cellular structure are Voronoi patterns (as these occur frequently due to physical constraints) [9]. Gómez-Gálvez et al. observed that the fundamental problem of packing occurs when the polygonal shapes at apical and basal surfaces do not match (e.g. pentagonal top and hexagonal bottom) leading to topological shift and resulting in scutoids.

The literature on this discovery shows the occurrence of scutoids and provides some statistical information of when and how they form [7, 2] (See 3). The reason why these shapes occur in nature is that they are the sole enablers for a space-filling packing on the skin cells. The scutoidal shapes can be considered as interpolation of 2D tiling patterns, that usually consists of hexagons and pentagons that appear on many natural structures. Interpolation is obtained by edge-collapse and vertex-split operations.



(a) *Two views of a scutoid.* (b) *Two views of a Delaunay loft.*

Figure 3: The comparison of scutoids with our Delaunay Lofts. The original scutoids usually depicted using straight edges as shown in this visual representation. Delaunay Lofts, on the other hand, (1) have curved edges and (2) can fill space.

The Figure 3a demonstrates a usual depiction of the originally discovered scutoid structures obtained by edge-collapse or vertex-split operations between pentagons and hexagonal faces. This view results in non-planar pentagons or hexagons with straight boundaries as shown in the Figure 3a, but it does not provide any well-defined process to fill inside of these non-planar faces. Our approach to obtaining scutoid-like structures is to use 3D Voronoi decomposition using a set of curves as Voronoi sites. If this set of curves are closed under symmetry operations, the resulted Voronoi shapes are guaranteed to be space filling. It is interesting to note that this approach is also in sync with Delaunay's original intention for the use of Delaunay diagrams. Delaunay was, in fact, the first to use symmetry operations on points (instead of curves) and Voronoi diagrams to produce space filling polyhedra, which he called Stereohedra [4, 12]. Our approach can be viewed as an extension of his idea to curves. We, therefore, called our approach Delaunay Lofting.

3 Methodology and Implementation

When using points, construction of 3D Voronoi decomposition is relatively simple since distances to points guarantee to produce planar faces. On the other hands, when we use curves or even straight lines Voronoi decomposition can produce curved faces, which, in fact, makes this method interesting. However, having curved faces significantly complicates the algorithms to construct 3D Voronoi decomposition in high resolution. We, therefore, choose to deal with a subset of this general problem.

We decompose thin rectangular structures that consist of a discrete set of z-constant planar layers. We also choose the control curves in the form of $(x_i = f_{i,x}(z), y_i = f_{i,y}(z))$, where $i = 0, 1, \dots, n$. This constraint guarantees that each curve intersects with each layer only once. We also use a specific distance function to further simplify the process into a set of 2D Voronoi decomposition. Based on these simplifications, the general process that consists of the following steps: (1) Discretize the rectangular prism with N number of constant z planes, which we call layers. (2) Design M number of curves inside of the rectangular domain. (3) Find the intersection of curves with intermediate layers. (4) For each layer, compute its Voronoi partitioning by using intersection points with that particular layer as Voronoi sites[1]. (5) Offset each Voronoi polygon the same amount using Minkowski difference [2]. (6) Treat each vertex as a single manifold and insert edges between consecutive vertices [3]. (7) Insert edges between closest vertices in consecutive layers based on face normal.

This process automatically creates the Delaunay Lofts and the resulting structures resemble scutoids with curved edges and faces. To produce space filling tiles, control curves must be closed under symmetry operations and each rectangle must be a regular domain, which topologically forms a 2-toroid. To easily produce control curves that are closed under symmetry operations, we interpolate 2D Delaunay diagrams with wallpaper symmetries. If the top and bottom tilings are rigid transformations of each other, this process produces 3D space filling shapes. In the 3D printed examples shown in Figures 1, 2 and, 4, control curves are just straight lines that interpolates top and bottom Delaunay vertices. We also have examples that interpolate more than two tiles, one such example is shown in Figure 3b, but we have not printed those yet.

References

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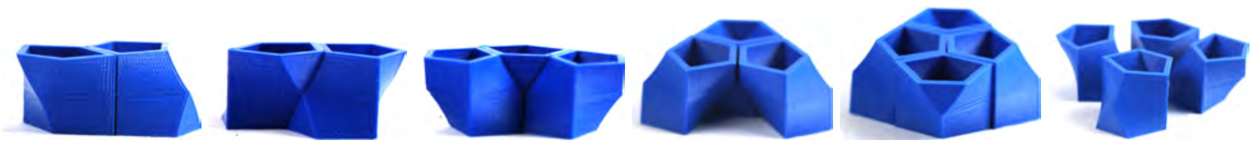
[1] Since space is bounded, the boundaries of the prism become part of Voronoi polygons. For regular domain, we compute Voronoi decomposition in a 2-toroidal domain.

[2] Note that this offsetting process can also change the topology of the polygons.

[3] This process turns each original face into a 2-sided face [14], which is actually a 2-manifold.



(a) *Two views of a two 3-Space Filling 2.5 Tiles.* (b) *Three tiles.* (c) *Three tiles.* (d) *Four tiles.* (e) *Four tiles separated.*

Figure 4: Another example of single space filling Delaunay Loft tile This tile is created as an interpolation of three layers of tilings, namely (1) a semi-regular pentagonal; (2) a regular rectangular and; (3) another semi-regular pentagonal tiling, which is rotated version of the first pentagonal tiling. In this case, the interpolating control curves are also straight lines. Regular rectangular tiling in the middle is again an automatically produced byproduct.

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